REVIEW ARTICLES

Recent advances in classification of finite-dimensional estimation algebras for nonlinear filtering systems*

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Received March 18, 1999; revised August 12, 1999

Abstract The recent progresses on Brockett's problem of classifying all finite-dimensional estimation algebras of maximal rank for nonlinear filtering systems are introduced.

Keywords: finite-dimensional filter, estimation algebra of maximal rank, nonlinear drift.

During the last decade, a number of progresses have been made on Brockett's problem of classification of finite-dimensional estimation algebras for nonlinear filtering systems. In the following, we review these progresses and expose some key points behind them.

1 Brockett's problem of classification

Consider the following signal observation model:

$$\begin{cases} \mathrm{d}x(t) = f(x(t))\mathrm{d}t + g(x(t))\mathrm{d}v(t), & x(0) = x_0, \\ \mathrm{d}y(t) = h(x(t))\mathrm{d}t + \mathrm{d}w(t), & y(0) = 0, \end{cases} \tag{1}$$

in which x, v, y, and w are $\mathbb{R}^n$-, $\mathbb{R}^n$-, $\mathbb{R}^m$-, and $\mathbb{R}^m$- valued processes respectively, and v and w are independent, standard Brownian processes. Suppose that the vector functions f and h are C^∞ smooth and $g(x)$ is an orthogonal matrix for each $x \in \mathbb{R}^n$. $x(t)$ is called the state of the system at time t and $y(t)$, the observation at time t. $\rho(t,x)$, the conditional probability density of the state $x(t)$, conditioned on the observation $\{y(s): 0 \leqslant s \leqslant t\}$, is determined by the Duncan-Mortensen-Zakai equation, which in the unnormalized form is given by

$$\mathrm{d}\rho(t,x) = L_0\rho(t,x)\mathrm{d}t + \sum_{i=1}^{m} L_i\rho(t,x)\mathrm{d}y_i(t), \quad \rho(0,x) = \rho_0(x), \tag{2}$$

where

$$L_0 = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^{n} f_i\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i} - \frac{1}{2}\sum_{i=1}^{m} h_i^2, \tag{3}$$

and for $i = 1, \cdots, m$, L_i is the function operator of multiplication by h_i, and ρ_0 the probability density

* Project supported by the National Natural Science Foundation of China (Grant No. 79790130).

of the initial state x_0. Here and/or here-in-after, the following notations are used.

(i) $\mathscr{P}_i$ is the totality of all the polynomials of degree less than or equal to i.

(ii) $\mathscr{H}_2$ is the totality of all the homogeneous polynomials of degree-two. We use the convention that $0 \in \mathscr{H}_2$. Whenever necessary, $\mathscr{H}_2(x_1, \cdots, x_r)$ is used to specify the dependent variables of $\mathscr{H}_2$. The default dependent variables are $x_1, \cdots, x_n$.

(iii) If $\boldsymbol{a}$ is a vector, the notation $\boldsymbol{a}_i$ stands for the i-th component of $\boldsymbol{a}$.

Define

$$D_i = \frac{\partial}{\partial x_i} - f_i \tag{4}$$

and

$$\eta = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 + \sum_{i=1}^{m} h_i^2. \tag{5}$$

Then

$$L_0 = \frac{1}{2}\left(\sum_{i=1}^{n} D_i^2 - \eta \right). \tag{6}$$

Definition 1. *The estimation algebra* $\mathscr{E}$ of the filtering system (1) is defined to be the Lie algebra generated by $\{L_0, L_1, \cdots, L_m\}$.

Ocone[1] observed the following basic property for an estimation algebra $\mathscr{E}$ to be finite-dimensional.

Theorem 1. *Let $\mathscr{E}$ be a finite-dimensional estimation algebra. If φ is a function in $\mathscr{E}$, then φ is a polynomial of degree less than or equal to two, i.e. $\varphi \in \mathscr{P}_2$.*

Finite-dimensional estimation algebras can be used to construct finite-dimensional nonlinear filters, which was initially proposed by Brockett and Clark[2], Brockett[3], and Mitter[4]. In his talk at the International Congress of Mathematicians in 1983, Brockett[5] proposed to classify all finite-dimensional estimation algebras. Since then, a number of progresses have been made on Brockett's problem of classification. In the following, we shall restrict ourselves to classifying estimation algebras of maximal rank.

Chiou and Yau[6] introduced the condition of maximal rank.

Definition 2. *An estimation algebra $\mathscr{E}$ is said to be of maximal rank if for every $1 \leqslant i \leqslant n$ there exists a constant c_i such that $x_i + c_i \in \mathscr{E}$.*

Let $\mathscr{E}_0$ be the real vector space spanned by $1, x_1, \cdots, x_n, D_1, \cdots, D_n$ and L_0. The following lemma is an immediate consequence of the maximal rank assumption.

Lemma 1. *Let $\mathscr{E}$ be an estimation algebra of maximal rank associated with the filtering system* (1). *Then $\mathscr{E} \supset \mathscr{E}_0$.*

If $\mathscr{E}$ is of maximal rank, Lemma 1 immediately gives that

$$\mathscr{E} \ni [D_j, D_i] = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j} =: \omega_{ij}, \quad 1 \leqslant i, j \leqslant n. \tag{7}$$

If further $\mathscr{E}$ is finite-dimensional, then Theorem 1 implies that $\omega_{ij} \in \mathscr{P}_2$ for $1 \leqslant i, j \leqslant n$.

2 Preliminaries

Wong[7] introduced the following concept of Ω-matrix:

$$\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1n} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1} & \omega_{n2} & \cdots & \omega_{nn} \end{pmatrix} = (\omega_{ij}). \tag{8}$$

Note that f_i, as a tensor field, is the differential 1-form $\sum_{i=1}^{n} f_i \mathrm{d}x_i$ and ω_{ij}, as a tensor field, is the differential 2-form $\sum_{1 \leqslant i < j \leqslant n} \omega_{ij} \, \mathrm{d}x_i \wedge \mathrm{d}x_j$. The exterior derivative of the former is just the latter, i.e. they have the following relation

$$\mathrm{d}\Big(\sum_{i=1}^{n} f_i \mathrm{d}x_i \Big) = \sum_{1 \leqslant i < j \leqslant n} \omega_{ij} \mathrm{d}x_i \wedge \mathrm{d}x_j. \tag{9}$$

On one hand, since $\mathrm{d}^2 = 0$, we deduce that ω_{ij}'s satisfy the cyclic relation:

$$\frac{\partial \omega_{ij}}{\partial x_l} + \frac{\partial \omega_{li}}{\partial x_j} + \frac{\partial \omega_{jl}}{\partial x_i} = 0, \quad 1 \leqslant i, j, l \leqslant n. \tag{10}$$

On the other hand, the Poincaré's Lemma means that every d-closed differential form in $\mathbb{R}^n$ is d-exact. Then, $\omega_{ij} \equiv 0$ for $1 \leqslant i$, $j \leqslant n$ means that f_i is d-closed and thus is d-exact, i.e. f_i is a gradient vector field. The following Theorem 2 of Yau[8] is a natural extension of the last assertion.

Theorem 2. *Suppose that Ω is a constant matrix. Then the drift term f must be a linear vector field (i.e. each component is a polynomial of degree one) plus a gradient vector field.*

The next two theorems were proved by Yau[8] and will be used in the classification of finite-dimensional estimation algebras of maximal rank for nonlinear filtering systems.

Theorem 3. *Let $F(x_1, \cdots, x_n)$ be a polynomial on $\mathbb{R}^n$. Suppose that there exists a polynomial path $c: \mathbb{R} \to \mathbb{R}^n$ such that $\lim_{t \to \infty} |c(t)| = \infty$, and $\lim_{t \to \infty} F(c(t)) = -\infty$. Then there are no C^∞*

functions $f_1, \cdots, f_n$ *on* $\mathbb{R}^n$ *satisfying the equation*

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = F. \tag{11}$$

Theorem 4. *Suppose that the estimation algebra* $\mathscr{E}$ *of* (1) *is both of finite dimension and of maximal rank and that* Ω *is a constant matrix. Then,* $h_i \in \mathscr{P}_1$ *for* $1 \leqslant i \leqslant m$, $\eta \in \mathscr{P}_2$, *and* $\mathscr{E} = \mathscr{E}_0$.

Proof. Since $\mathscr{E}$ is of maximal rank, $D_i \in \mathscr{E}$ by Lemma 1. Then,

$$\mathscr{E} \ni [L_0, D_i] = \sum_{j=1}^{n} \omega_{ij} D_j + \frac{1}{2} \frac{\partial \eta}{\partial x_i}.$$

While ω_{ij}'s are constants, we obtain

$$\frac{\partial \eta}{\partial x_i} \in \mathscr{E}.$$

In view of Theorem 1,

$$\frac{\partial \eta}{\partial x_i} \in \mathscr{P}_2,$$

and thus $\eta \in \mathscr{P}_3$. According to the eq. (5),

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} + \sum_{i=1}^{n} f_i^2 = \eta - \sum_{i=1}^{n} h_i^2 \tag{12}$$

which immediately implies the desired results by Theorem 3. Q.E.D.

If Ω is a constant matrix, then Theorems 2 and 4 immediately imply the three assertions of the classification theorem (see Theorem 6 in section 3 below), respectively. The proof of the classification theorem is reduced to show that Ω is a constant matrix.

Recall that $\mathscr{H}_2$ is the space of quadratic forms in n variables, i.e. the real vector space spanned by $x_i x_j$, with $1 \leqslant i \leqslant j \leqslant n$. Let $X = (x_1, \cdots, x_n)'$.

Definition 3. *For any quadratic form* $p \in \mathscr{H}_2$, *there exists a symmetric* $n \times n$ *matrix* A *such that* $p(x) = X'AX$. *The rank of the quadratic form* p *denoted by* $r(p)$ *is defined to be the rank of the matrix* A.

Definition 4. *A fundamental quadratic form of the estimation algebra* $\mathscr{E}$ *is an element* $p_0 \in \mathscr{E} \cap \mathscr{H}_2$ *with the biggest positive rank, i.e.* $r(p_0) \geqslant r(p)$ *for any* $p \in \mathscr{E} \cap \mathscr{H}_2$. *The quadratic rank of the estimation algebra* $\mathscr{E}$ *is defined to be the rank of a fundamental quadratic form of* $\mathscr{E}$.

Let p_0 be a fundamental quadratic form of $\mathscr{E}$ and $k := r(p_0)$. After an orthogonal transformation on x, p_0 can be written as

$$p_0 = \sum_{i=1}^{k} c_i x_i^2, \quad c_i \neq 0. \tag{13}$$

Chen and Yau[9] proved the following Theorem 5.

Theorem 5. *Let $\mathscr{E}$ be a finite-dimensional estimation algebra of maximal rank. Let k be the quadratic rank of $\mathscr{E}$, and p_0 (defined by eq. (13)) a fundamental quadratic form of $\mathscr{E}$. Then $p \in \mathscr{E} \cap \mathscr{H}_2$ implies that p depends only on the k variables $x_1, x_2, \cdots, x_k$, i.e. $p \in \mathscr{H}_2(x_1, \cdots, x_k)$.*

3 Recent advances

We begin with introducing the notation:

(i) β_{ij} is the homogeneous polynomial of the degree-one part of ω_{ij} (if it exists, if not it is zero), that is, β_{ij} is a linear combination of $x_1, \cdots, x_n$ with no constant term.

(ii) $A_r(i, j)$ is the coefficient of x_r in β_{ij}, and A_r is the matrix in which the (i, j)-component is $A_r(i,j)$. Thus

$$A_r(i,j) = \frac{\partial \beta_{ij}}{\partial x_r}, \quad \beta_{ij} = \sum_{r=1}^{n} A_r(i, j) x_r. \tag{14}$$

(iii) η_4 is the homogeneous polynomial of the degree-4 part of η (where $\eta \in \mathscr{P}_4$).

(iv) Decompose the $\boldsymbol{\Omega}$-matrix into blocks in the manner of

$$\boldsymbol{\Omega} = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ -\Omega_{12}' & \Omega_{22} \end{pmatrix}$$

with $\boldsymbol{\Omega}_{11} \in \mathbb{R}^{k\times k}$, $\boldsymbol{\Omega}_{12} \in \mathbb{R}^{k\times(n-k)}$ and $\boldsymbol{\Omega}_{22} \in \mathbb{R}^{(n-k)\times(n-k)}$.

Write

$$\begin{aligned} \sum_{i=1}^{n} \beta_{ji}\beta_{li} - \frac{1}{2}\frac{\partial^2 \eta_4}{\partial x_l \partial x_j} &:= C(j,l) := \sum_{r,p=1}^{n} C_{rp}(j,l) x_r x_p \\ &= \sum_{r=1}^{n} C_{rr}(j,l) x_r^2 + 2\sum_{1\leq r<p\leq n} C_{rp}(j,l) x_r x_p \\ &= \sum_{r=1}^{k} C_{rr}(j,l) x_r^2 + 2\sum_{1\leq r<p\leq k} C_{rp}(j,l) x_r x_p \in \mathscr{E} \cap \mathscr{H}_2(x_1, x_2, \cdots, x_k) \end{aligned} \tag{15}$$

with

$$C_{rp}(j,l) = C_{pr}(j,l); \qquad C_{rp}(j,l) = 0 \text{ if } r > k \text{ or } p > k. \tag{16}$$

Tam, Wong and Yau[10] classified all finite dimensional exact estimation algebras of maximal rank with arbitrary state-space dimension. Chiou and Yau[6] introduced the concept of maximal rank estimation algebras and classified all finite-dimensional estimation algebras of maximal rank with state-space dimension less than or equal to two. The novelty of their theorem is that there is no assumption on the drift term of the nonlinear filtering system, and the proof lied in proving that ω_{12} is a constant (note that $\omega_{11}=0$ and $\omega_{22}=0$). Later, Chen, Yau and Leung[11] improved Chiou and Yau's result in that the dimension of the state-space was assumed to be less than or equal to three, and their proof consisted of proving that the three entries ω_{12}, ω_{13} and ω_{23} are constants.

In 1996, Chen and Yau[9] began to study the $\boldsymbol{\Omega}$-matrix in a systematical way. They concluded that $\omega_{ij}\in \mathscr{P}_1$ for $1\leqslant i,\ j\leqslant n$ and further that $\boldsymbol{\Omega}_{11}$ is a constant matrix. Later, Chen and Yau[12] proved that $\boldsymbol{\Omega}_{12}$ is a constant matrix.

By showing that $\boldsymbol{\Omega}_{22}$ is a constant matrix, Chen, Yau and Leung[13] could classify all finite-dimensional estimation algebras of maximal rank when the state space dimension is less than or equal to four. Their proof is sketched as follows. They first noticed that the relation $[[L_0,\ D_j],\ D_l]\in \mathscr{E}$ implies the following

$$\sum_{i=1}^{n}\beta_{ji}\beta_{li}-\frac{1}{2}\frac{\partial^2\eta_4}{\partial x_l\partial x_j}\in \mathscr{E}\cap \mathscr{H}_2(x_1,\ \cdots,\ x_k) \tag{17}$$

and then for $p,j,l=1,\ \cdots,\ n$ and $q=k+1,\cdots,n$,

$$\sum_{i=1}^{n}[A_p(j,i)A_q(l,i)+A_q(j,i)A_p(l,i)]=\frac{1}{2}\frac{\partial^4\eta_4}{\partial x_j\partial x_q\partial x_p\partial x_l}. \tag{18}$$

Further, the relation (18) implies

$$\sum_{i=1}^{n}A_j(l,i)A_j(i,l)=\sum_{i=1}^{n}A_l(j,i)A_l(i,j)$$

$$=\frac{1}{2}\sum_{i=1}^{n}(A_j(j,i)A_l(i,l)+A_l(j,i)A_j(i,l)),\quad j,l=k+1,\cdots,n, \tag{19}$$

and

$$\sum_{i=1}^{n}A_j(j,i)A_j(i,l)=\sum_{i=1}^{n}A_j(j,i)A_l(i,j),\quad j=k+1,\cdots,n,\quad l=1,\cdots,n. \tag{20}$$

From (19), Chen, Yau and Leung[13] derived

$$2\sum_{i=1}^{n}(A_j(l,i))^2+2\sum_{i=1}^{n}(A_l(j,i))^2=2\sum_{i=1}^{n}A_j(j,i)A_l(l,i)+2\sum_{i=1}^{n}A_j(l,i)A_l(j,i), \tag{21}$$

and using the Schwarz inequality, they obtained

$$2\sum_{i=1}^{n}(A_j(l,i))^2+2\sum_{i=1}^{n}(A_l(j,i))^2$$

$$\leqslant \sum_{i \neq j, l} \{ (A_j(j,i))^2 + (A_l(l,i))^2 + (A_j(l,i))^2 + (A_l(j,i))^2 \}. \tag{22}$$

By taking the sum of both sides of (22) over $j < l$, they got

$$\sum_{i \neq j, i \neq l} (A_j(l,i))^2 \leqslant (n-4) \sum_{i \neq j} (A_j(j,i))^2. \tag{23}$$

At this stage, by assuming that $n \leqslant 4$, they derived from (23) that

$$A_j(l,i) = 0 \qquad \text{for } i \neq j,\ i \neq l, \quad j, l = k+1, \cdots, n, \tag{24}$$

and then proved that Ω_{22} is a constant matrix for $n \leqslant 4$.

To get (24) from (23), it is essential to assume that $n \leqslant 4$. The arguments of Chen, Yau and Leung[13] strongly depend on the assumption that the state dimension $n \leqslant 4$, and thus are difficult to be generalized to the general case of arbitrary state dimension.

Recently, the author[1)] got around the above difficulty, and classified all finite-dimensional estimation algebras of maximal rank with arbitrary state-space dimension. The author[1)] proved the following general classification theorem.

Theorem 6. *Let $\mathscr{E}$ be a finite-dimensional estimation algebra of the filtering problem* (1) *of maximal rank, and $\mathscr{E}_0$ the real vector space of dimension $2n+2$ with basis given by $1, x_1, \cdots, x_n, D_1, \cdots, D_n$ and L_0. Then,*

(i) *the drift term f must be a linear vector field* (*i.e. each component is a polynomial of degree less than or equal to one*) *plus a gradient vector field*;

(ii) *η is a polynomial of degree less than or equal to two*;

(iii) $\mathscr{E} = \mathscr{E}_0$.

Theorem 6 improves both results of Tam, Wong and Yau[10] and Chen, Yau and Leung[13] in that Theorem 6 neither assumes that the finite-dimensional estimation algebra under consideration is exact nor assumes that the state-space dimension is less than or equal to four.

Mitter once conjectured that all functions in finite-dimensional estimation algebra $\mathscr{E}$ are linear in x. It is obvious that the second assertion of the classification theorem implies the Mitter conjecture for the case of estimation algebras of maximal rank.

The key points of the author's proof of Theorem 6 are the following two propositions:

Proposition 1. *We have for $j, l = k+1, \cdots, n$,*

1) Tang, S., Brockett's problem of classification of finite-dimensional estimation algebras for nonlinear filtering systems, submitted to *SIAM J. Control & Optim*. Also in *Abstracts of Short Communications and Poster Sessions*, *International Congress of Mathematicians* (Berlin, August 18—27, 1998), p. 352.

$$\sum_{i=1}^{n}(A_j(l,i))^2+\sum_{i=1}^{n}(A_l(j,i))^2+\sum_{i=1}^{n}(A_j(l,i)-A_l(j,i))^2$$

$$=2\sum_{i=1}^{n}A_j(j,i)A_l(l,i). \tag{25}$$

Proposition 2. *We have*

$$A_j(j,i)=0 \text{ for } j=k+1,\cdots,n \text{ and } i=1,\cdots,n. \tag{26}$$

Propositions 1 and 2 immediately imply that Ω_{22} is a constant matrix, and the proof of Theorem 6 is then finished.

Proposition 1 is easily seen from (19). To prove Proposition 2, the author introduced a series of new computations about the estimation algebra $\mathscr{E}$. Fig. 1 outlines the computation routine.

$$\begin{tikzcd}
 & U_i \arrow[d, equal] & V_j \arrow[d, equal] & W_j \arrow[d, equal] \\
D_j \arrow[r, "L_0"] & {[L_0,D_j]} \arrow[r, "L_0"] \arrow[d, "D_l"] & {[L_0,[L_0,D_j]]} \arrow[r, "L_0"] \arrow[d, "D_j"] & {[L_0,[L_0,[L_0,D_j]]]} \arrow[d, "D_j"] \\
 & {[U_j,D_l]} & {[V_j,D_j]} \arrow[d, "D_l"] & {[W_j,D_j]} \arrow[d, "D_j"] \\
 & & {[[V_j,D_j],D_l]} & {[[W_j,D_j],D_j]} \arrow[d, "D_j"] \\
 & & & {[[[W_j,D_j],D_j],D_j]}
\end{tikzcd}$$

Fig. 1. Computation chart

The last two columns of computations in the chart are completely new, and their last terms $[[V_j, D_j], D_l]$ and $[[[W_j, D_j], D_j], D_j]$ turn out to have the same form: "a homogeneous polynomial of degree-two" + "an element of $\mathscr{E}_0$".

From these two terms, the author obtained the two homogeneous polynomials in $\mathscr{E}$ of degree-two:

$$\sum_{i,r=1}^{n}[A_r(j,i)\beta_{jr}\beta_{li}+A_i(j,r)\beta_{jr}\beta_{li}-A_j(j,i)\beta_{lr}\beta_{ri}$$

$$-A_l(j,i)\beta_{jr}\beta_{ri}-\beta_{ji}A_l(j,r)\beta_{ri}-\beta_{ji}\beta_{jr}A_l(r,i)]$$

$$+\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_j}(C(j,i))\beta_{li}-A_j(j,i)C(l,i)\right)\in\mathscr{E}\cap\mathscr{H}_2 \tag{27}$$

and

$$\begin{aligned}
&-\sum_{i,l,r=1}^{n}[A_l(j,i)\beta_{ir}\beta_{jl}+A_i(j,r)\beta_{il}\beta_{jl}+A_r(j,i)\beta_{il}\beta_{jl}+A_i(j,l)\beta_{ir}\beta_{jl}]A_j(j,r)\\
&-2\sum_{i,l,r=1}^{n}\frac{\partial}{\partial x_j}(A_l(j,i)\beta_{ir}\beta_{jl}+A_i(j,r)\beta_{il}\beta_{jl}+A_r(j,i)\beta_{il}\beta_{jl}+A_i(j,l)\beta_{ir}\beta_{jl})\beta_{jr}\\
&+\sum_{l,r=1}^{n}\left(\frac{\partial}{\partial x_l}(C(j,r))\beta_{jl}+\frac{\partial}{\partial x_r}(C(j,l))\beta_{jl}\right)A_j(j,r)\\
&+2\sum_{l,r=1}^{n}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_l}(C(j,r))\beta_{jl}+\frac{\partial}{\partial x_r}(C(j,l))\beta_{jl}\right)\beta_{jr}\\
&-2\sum_{i,l,r=1}^{n}[A_l(j,i)A_j(i,r)+A_i(j,r)A_j(i,l)]\beta_{jl}\beta_{jr}+4\sum_{l,r=1}^{n}C_{jl}(j,r)\beta_{jl}\beta_{jr}\\
&-3\sum_{i,r=1}^{n}\frac{\partial}{\partial x_j}[C(j,i)\beta_{ir}]A_j(j,r)-3\sum_{i,r=1}\frac{\partial^2}{\partial x_j^2}[C(j,i)\beta_{ir}]\beta_{jr}\\
&-3\sum_{i,r=1}^{n}\Big\{2A_r(j,i)\Big[\sum_{l=1}^{n}\beta_{jl}\beta_{li}+C(j,i)\Big]+A_j(j,i)\Big[\sum_{l=1}^{n}\beta_{rl}\beta_{li}+C(r,i)\Big]\\
&+\beta_{ji}\frac{\partial}{\partial x_j}\Big[\sum_{l=1}^{n}\beta_{rl}\beta_{li}+C(r,i)\Big]+A_i(j,r)\Big[\sum_{l=1}^{n}\beta_{jl}\beta_{li}+C(j,i)\Big]\Big\}A_j(j,r)\\
&-3\sum_{i,r=1}^{n}\frac{\partial}{\partial x_j}\Big(2A_r(j,i)\Big[\sum_{l=1}^{n}\beta_{jl}\beta_{li}+C(j,i)\Big]+A_j(j,i)\Big[\sum_{l=1}^{n}\beta_{rl}\beta_{li}+C(r,i)\Big]\\
&+\beta_{ji}\frac{\partial}{\partial x_j}\Big[\sum_{l=1}^{n}\beta_{rl}\beta_{li}+C(r,i)\Big]+A_i(j,r)\Big[\sum_{l=1}^{n}\beta_{jl}\beta_{li}+C(j,i)\Big]\Big)\beta_{jr}\\
&+\frac{1}{2}\sum_{i=1}^{n}\frac{\partial^3}{\partial x_j^3}\left(C(j,i)\frac{\partial\eta_4}{\partial x_i}\right)\in\mathscr{E}\cap\mathscr{H}_2.
\end{aligned}\tag{28}$$

Suppose that $j\geqslant k+1$. The coefficient of x_j^2 in the first polynomial is equal to

$$-\sum_{i,r=1}^{n}A_j(j,r)A_r(j,i)A_j(i,l)-A_jA_lA_j(j,j), \tag{29}$$

while in the second polynomial, the coefficient of x_j^2 is equal to

$$-6\sum_{i,l,r=1}^{n}A_j(j,l)A_l(j,i)A_j(i,r)A_j(r,j)+24A_j^4(j,j). \tag{30}$$

These coefficients should be zero as $j \geqslant k+1$ by Theorem 5. In this way, the author established the two sets of new equations about the Ω-matrix:

$$\sum_{i,r=1}^{n} A_j(j,r) A_r(j,i) A_j(i,l) = -A_j A_l A_j(j,j) = 0, \qquad j = k+1, \cdots, n, \quad l = 1, \cdots, n \tag{31}$$

and

$$A_j^4(j,j) = \frac{1}{4} \sum_{i,l,r=1}^{n} A_j(j,l) A_l(j,i) A_j(i,r) A_j(r,j), \quad j = k+1, \cdots, n. \tag{32}$$

Putting (31) into (32), the author obtained

$$A_j^4(j,j) = 0, \quad j = k+1, \cdots, n. \tag{33}$$

Since the matrix A_j is skew symmetric and A_j^2 symmetric, the author further had

$$A_j^2(j,l) = 0, \quad j = k+1, \cdots, n, \quad l = 1, \cdots, n \tag{34}$$

and thus Proposition 2 is proved.

There are many other striking works related with this paper, among which are Cohen de Lara[14, 15], Davis and Marcus[16], Dong, Tam, Wong et al.[17], Duncan1), Marcus[18], and Wong[19, 20]. The recent perspective paper1) provides a good review of the past and the present of the filtering theory.

Acknowledgement The author would like to thank Professors Tyrone E. Duncan and Li Xunjing for their helpful discussions.

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